Constructions of Grassmannian Simplices

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Abstract

The aim of this article is to present new and explicit constructions of optimal packings in the Grassmannian space. Therefore we use a method presented in [15] involving finite group representations. Infinite families of configurations having only one non-trivial set of principal angles are naturally found using 2-transitive groups. These packings are proved to reach the simplex bound [17] and are therefore optimal w.r.t. the chordal distance. The construction is illustrated by several examples.

Keywords : Grassmannian packing, Space-time coding, Simplex bound, Wireless communication.

1   Introduction

Space time codes have been fiercely studied since the publication of [14] and [13] where the ability of such codes to enhance communications is revealed. Part of this interest has turned over Grassmannian coding when the link between the non-coherent case and Grassmannians was clearly explained by Zheng and Tse in [22]. Several papers (as [18]) have shown that if the so-called product distance is linked to the performance of the code at high SNR (Signal to Noise Ratio), the chordal distance is a key parameter for performance at low SNR . Since the codes described in this article are optimal regarding the chordal distance as they meet the simplex bound introduced in [17], one may expect them to enhance communications especially when the channel is very noisy. The notion of chordal distance has first appeared in [17] where general constructions for Grassmannian packings are described. We were particularly interested in the following classical construction using group orbits. Let W be a m dimensional subspace in $\mathbb{C}^n$ and let G be a finite subgroup of $U_n(\mathbb{C})$ then we can consider the Grassmannian code made of the orbit of W under G.

Constructions using group orbits have motivated coding theorists for years and have already been successfully used in the context of spherical codes or codes over finite fields. In the Grassmannian case as others the result of such a construction strongly depends on the group and the first element of the orbit. This choice arises then as the main problem.

There are two kind of codes obtained as the orbit of an element under a group G. Codes having a cardinality equal to the cardinality of the group and codes where the stabilizer of an element is a non-trivial subgroup of G. This second kind of codes are naturally more structured and smaller. The originality of this article is to take the problem in a unusual way. We first search in the group G which subgroups are interesting candidates for a stabilizer then we deduce a convenient starting element.

To compare Grassmannian codes obtained by various ways, we introduce the parameters of a Grassmannian configurations as the numbers $[n, m, N, d]$, where m is the dimension of the subspaces, n the dimension of the surrounding space, N the number of elements, and d the minimal (chordal) distance. In this article we compute the parameters of Grassmannian packings obtained by the use of 2-transitive groups and subspaces coming from representation theory and show that they meet the so-called simplex bound (Thm 7) providing optimal packings.

We first recall in Section 2 basic facts concerning Grassmannian spaces and representation theory. Then we present how to use 2-transitive groups and representation theory to construct special configurations which reach the simplex bound. Section 3 gives a detailed formulation of this result and its proof. Explicit examples coming from the classification of 2-transitive groups are developed in Section 4. We end with some generalizations in Section 5 and conclude.

2 Basic facts on Grassmannian spaces and representation theory

2.1 Grassmannian spaces

In this section we recall from $[17]$[1] the main background about Grassmannian spaces used in the sequel. We introduce Grassmannian spaces, define distances between elements and recall the expression of the simplex bound.

The Grassmannian space over the complex numbers denoted by $\mathcal{G}_{m,n}$ is simply the set of all m-dimensional vector subspaces of $\mathbb{C}^n$. To introduce a distance between subspaces the notion of principal angles is needed.

Definition 1 *For any $\mathcal{P}, \mathcal{Q}$ elements of $\mathcal{G}_{m,n}$ (i.e. two m-dimensional subspaces in $\mathbb{C}^n$), let*[2]

$$x_1, y_1 := arg \max_{\substack{x \in \mathcal{P}, y \in \mathcal{Q} \\ ||x||=||y||=1}} |\langle x, y \rangle|.$$

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[1]Warning : their notations for n and m are different from ours.

[2] By "arg max" we mean the arguments (any) which allow the following function to reach it's maximum.

Then by induction, constrain x_i (resp. y_i) to be a unit vector orthogonal to $\{x_1, \ldots, x_{i-1}\}$ (resp. $\{y_1, \ldots, y_{i-1}\}$) and define

$$x_i, y_i := arg \max_{\substack{x \in \mathcal{P}, y \in \mathcal{Q} \\ constrained}} |\langle x, y \rangle|, \quad i \leq m.$$

The principal angles are the values $\theta_i \in [0, \frac{\pi}{2}]$ such that $\cos(\theta_i) = |\langle x_i, y_i \rangle|$, $i = 1, \ldots, m$.

One can prove that the set of principal angles $(\theta_1, \ldots, \theta_m)$ characterizes the orbit of the pair $(\mathcal{P}, \mathcal{Q})$ under the action of the orthogonal group. To define distance between pairs we consider the following definitions :

$$d_c(\mathcal{P}, \mathcal{Q}) \quad := \sqrt{\sum_i \sin^2(\theta_i)}$$

$$\tilde{d}(\mathcal{P}, \mathcal{Q}) \quad := \prod_i \sin(\theta_i).$$

In this paper we mainly focus on the chordal distance d_c but we also refer to $\tilde{d}$. It is not a distance in the mathematical sense, but it is the key criterion for estimating the performance of a wireless communication at high SNR [18].

The chordal distance can also be expressed in an easy way using trace and projection matrices [17]. Let $\Pi_{\mathcal{P}}$ and $\Pi_{\mathcal{Q}}$ be the projection matrices on $\mathcal{P}$ and $\mathcal{Q}$ then,

$$d_c^2 = \frac{1}{2} ||\Pi_{\mathcal{P}} - \Pi_{\mathcal{Q}}||_2^2 = \text{Trace}(\Pi_{\mathcal{P}}) - \text{Trace}(\Pi_{\mathcal{P}}\Pi_{\mathcal{Q}}). \tag{1}$$

The chordal distance has another advantage. As shown in [17], this metric enables an embedding of the Grassmannian space in a sphere of higher dimension. One can then deduce bounds for codes in Grassmannian spaces from bounds for spherical codes. We recall here the *simplex bound* on Grassmannian configuration (obtained by this very way). This bound was stated for real Grassmannian spaces but extends itself in an easy way to the complex case[3].

Lemma 2 *[Simplex Bound] [17] For any configuration of N subspaces of dimension m in $\mathbb{C}^n$, the following inequality holds :*

$$d_c^2 \leq \frac{m(n-m)}{n} \frac{N}{N-1} \tag{2}$$

equality requiring $N \leq \binom{n+1}{2}$.

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[3]Indeed any complex configuration can be embedded into a real space doubling m, n and d_c^2. After simplifications one obtains the same bound.

If equality in (2) occurs then the distance between each pair of distinct elements is the same [17]. As an extension from the spherical case the term *simplicial* is used for such a configuration. One has to be careful since it is not true that any *simplicial* configuration meet the simplex bound (2). This paper deals with a particular case of simplicial configurations having only one non-trivial set of principal angles between any pairs of m-dimensional planes. We use the term *strongly simplicial* to denote configurations having only one non-trivial set of principal angles[4].

Next section recalls results of representation theory used in the sequel and sets some notations.

2.2 Representation theory

The constructions we want to study are based on representation theory. Some good references on the subject are [4–7, 10, 11] where one can find more details. We give here a brief summary of results and definitions.

Definition 3 *A unitary (complex) matrix representation of a finite group G is a morphism, $\rho : G \to U_n(\mathbb{C})$. The dimension of the matrices is called the dimension or the degree of the representation. The function $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(g) = Trace(\rho(g))$ is the character associated to ρ. It is worth noticing that $\chi_\rho(1)$ is equal to the dimension of ρ and that χ_ρ is constant on the conjugacy classes of G.*

Two representations ρ, ρ' are said to be equivalent if and only if there exists an invertible matrix U such that $\forall g \in G$, $U.\rho(g).U^{-1} = \rho'(g)$. As a consequence, two equivalent representations have equal characters. Moreover, one can show that conversely two representations having the same character are equivalent.

A representation ρ is called reducible if there exists a proper subspace W ($0 \subsetneq W \subsetneq \mathbb{C}^n$) such that $\forall g \in G$, $\rho(g)W \subset W$. It is irreducible otherwise, this vocabulary extends to characters[5]. When a representation ρ is reducible, the $\mathbb{C}$-vector space $\mathbb{C}^n$ can be decomposed as a direct sum of orthogonal subspaces isomorphic to irreducible representations (Maschke Theorem [5, 6]). This decomposition is not unique but the number of subspaces isomorphic to a given irreducible representation ρ_0 is well defined and called the multiplicity of ρ_0 in ρ. The direct sum of all subspaces isomorphic to a given irreducible representation is called an isotypic subspace and moreover the decomposition of $\mathbb{C}^n$ in isotypic subspaces is unique [5,6]. An isotypic subspace W is naturally characterized by an irreducible character χ of G and has this projection matrix [5,6]:

$$\Pi_W = \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)}\rho(g). \tag{3}$$

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[4]Similarly such configurations do not automatically reach the simplex bound.
[5]The character of an irreducible representation is called an irreducible character.

There are as many (non-equivalent) irreducible characters of G as conjugacy classes in G. The set of irreducible characters $\{\chi_i : i = 1, \ldots, t\}$, have the following properties (the so-called orthogonality relations) :

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \left\{ \begin{array}{l} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{array} \right. ,$$

$$\frac{1}{|G|} \sum_{i=1}^{t} \chi_i(g) \overline{\chi_i(h)} = \left\{ \begin{array}{l} |\mathrm{Cl}_G(g)| \text{ if } \mathrm{Cl}_G(g) = \mathrm{Cl}_G(h) \\ 0 \text{ if } \mathrm{Cl}_G(g) \neq \mathrm{Cl}_G(h) \end{array} \right. , \tag{4}$$

where $\mathrm{Cl}_G(g)$ denotes the conjugacy class of g in G, and $|.|$ is a notation for cardinality.

As a direct consequence, the set of irreducible characters forms an orthogonal basis of the space of class-functions[6] regarding the inner product :

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

The restriction of an irreducible character χ_ρ of G to a subgroup $H \subset G$ is noted $\chi_\rho \downarrow_H^G$. This new character may now be reducible and the representation $\rho \downarrow_H^G$ contains irreducible representations with multiplicities. If these multiplicities are denoted λ_i we may write

$$\chi \downarrow_H^G = \sum \lambda_i \chi_i$$

In order to avoid confusion in the sequel we prefix words with the group we want refer to (we use for example the term "H-irreducible" for representations irreducible w.r.t the group H, "H-isotypic" for isotypic subspaces w.r.t. the group H,...).

To lighten expressions when the context allows no misleading we note gW instead of $\rho(g)W$ for the action of $\rho(g)$ on the subspace W.

3   Code construction and optimality

The main result of this paper is the construction of *strongly simplicial* packings in Grassmannian spaces which reach the simplex bound and are therefore optimal w.r.t. the chordal distance. In this section we describe the construction and its elementary properties then we prove that 2-transitive groups acting on direct sums of isotypic subspaces give optimal configurations regarding the chordal distance.

The optimal packings we want to discuss come from the following construction given in [17].

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[6]A class function is a function which is constant on conjugacy classes.

Definition 4 *Let $\rho : G \to U_n(\mathbb{C})$ be an irreducible representation and H a subgroup of G. Take a non trivial subspace $W \subset \mathbb{C}^n$ of dimension m which is stable under the action of $\rho(H)$. We denote by $\mathcal{C}(W, G)$ the orbit of W under the action of G. The cardinality of this code is $|G/H'|$, where H' is the stabilizer of W in G.*

Without loss of generality we can assume that H is the stabilizer of W in G. Then the code $\mathcal{C}(W, G)$ has the nice following property :

Proposition 5 *The number of different sets of principal angles in a code $\mathcal{C}(W, G)$ is bounded above by the number of orbits for the action of G on unordered pairs of G/H. Moreover the set of principal angles for the pairs $(W, g_0 W)$ and (W, gW) is the same for all $g \in Hg_0 H \sqcup Hg_0^{-1}H$.*

Proof : Let $\mathrm{SPA}(W_1, W_2)$ be the set of principal angles between two subspaces W_1 and W_2. Since a unitary transformation preserves the set of principal angles (see Section 2.1) it is easy to see that for any g, g_1, g_2 in G and any h_1, h_2 in H,

$$\mathrm{SPA}(g_1 W, g_2 W) \;\; = \;\; \mathrm{SPA}(gg_1 h_1 W, gg_2 h_2 W) \;\; = \;\; \mathrm{SPA}(W, h_1^{-1}g_1^{-1}g_2 h_2 W).$$

The proposition follows directly.

After this proposition it is natural to look at groups G and subgroups H such that there are few double classes $H\backslash G/H$. The codes obtained with such groups have few distinct values for the distances between subspaces. This is true for any 'distance' definition which depends on the set of principal angles in particular for d_c and $\tilde{d}$.

Corollary 6 *If G is 2-transitive on G/H (or otherwise stated : if $|H\backslash G/H| = 2$) then there is only one non-trivial set of principal angles.*

Using group orbits to construct codes is a standard method. In the Grassmannian case as others the result depends strongly on the choices of starting element(s) and groups. In this context the choice of isotypic subspaces and 2-transitive groups is quite interesting as shows the following theorem.

Theorem 7 *Let G be a group and H a subgroup such that G acts 2-transitively by left multiplication on G/H. Suppose furthermore that we dispose of an irreducible representation $\rho : G \to GL_n(\mathbb{C})$ which is reducible when restricted to H. Under these hypothesis let W be a direct sum of H-isotopic subspaces[7], then the*

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[7] By H-isotypic subspaces we mean isotypic subspaces relative to the restriction of ρ to the subgroup H.

orbit of W under G forms a strongly simplicial[8] *Grassmannian configuration reaching the simplex bound. The parameters are $N = |G/H|$; $n = \dim(\rho)$ and $m = dim(W)$.*

If G is 2-transitive on G/H then H is a maximal subgroup of G. So the cardinality of the code $\mathcal{C}(W, G)$ is equal to $|G/H|$. The only result left to prove is that if W is a direct sum of isotypic subspaces then the simplex bound is reached. For this we need three lemmas. The first gives another expression of the value of the chordal distance in our constructions. The two others deal with sums related to character theory.

Lemma 8 *Let W be the direct sum of isotypic subspaces, associated to a subset of H-irreducible characters $\{\chi_1, ..., \chi_s\}$. Then the square value of the chordal distance between W and gW is*

$$d_c^2 = \sum_{i=1}^{s} \lambda_i \chi_i(1) - \frac{1}{|H|^2} \sum_{h_1, h_2 \in H} E(h_1)E(h_2)\chi_\rho(h_1 g h_2 g^{-1}) \qquad (5)$$

where

$$E(h) := \left(\sum_{i=1}^{s} \chi_i(1)\overline{\chi_i(h)} \right)$$

and λ_i is the multiplicity of χ_i in the restriction of χ_ρ to H (see Section 2.2) :

$$\chi_\rho \downarrow_H^G = \sum_{i} \lambda_i \chi_i.$$

Proof : The projection matrices on W and on gW are

$$\Pi_W = \sum_{i=1}^{s} \frac{\chi_i(1)}{|H|} \sum_{h \in H} \overline{\chi_i(h)}\rho(h),$$

$$\Pi_{gW} = \rho(g)(\Pi_W)\rho(g^{-1}) = \sum_{i=1}^{s} \frac{\chi_i(1)}{|H|} \sum_{h \in H} \overline{\chi_i(h)}\rho(ghg^{-1}).$$

The result follows from these expressions, formula (1) for chordal distance and the orthogonality relations (4). Indeed we have

$$\text{Trace}(\Pi_W) = \sum_{i=1}^{s} \lambda_i \chi_i(1)$$

and

$$\text{Trace}(\Pi_W \Pi_{gW}) = \frac{1}{|H|^2} \sum_{h_1, h_2 \in H} E(h_1)E(h_2)\text{Trace}(\rho(h_1)\rho(gh_2 g^{-1})).$$

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[8]We recall that by this we mean that there is only one non-trivial set of principal angles between any pair of distinct elements.

We now give two character formulas. We also need to introduce further notations.

Notations 9 *For any set S, the formal sum of all elements in S is written $\widehat{S}$. For a character $\chi(\widehat{S})$ means $\sum_{s \in S} \chi(s)$. As a consequence if χ is a character of G then $\chi(\widehat{Cl_G(h_1)}) = |Cl_G(h_1)|\chi(h_1)$.*

Lemma 10 *For any irreducible character χ_ρ on G,*

$$\chi_\rho\left(\widehat{Cl_G(h_1)}\,\widehat{Cl_G(h_2)}\right) = \frac{|Cl_G(h_1)||Cl_G(h_2)|\chi_\rho(h_1)\chi_\rho(h_2)}{\chi_\rho(1)}. \tag{6}$$

Proof : The following relation can be found in [6], chapter 30. If $\mathrm{Cl}_1, ..., \mathrm{Cl}_\ell$ are all conjugacy classes in G then

$$\widehat{\mathrm{Cl}_i}\,\widehat{\mathrm{Cl}_j} = \sum_{k=1}^{\ell} a_{ijk}\widehat{\mathrm{Cl}_k}$$

where

$$a_{ijk} = \frac{|\mathrm{Cl}_i||\mathrm{Cl}_j|}{|G|} \sum_{\chi \text{ irred.}} \frac{\chi(\mathrm{Cl}_i)\chi(\mathrm{Cl}_j)\overline{\chi(\mathrm{Cl}_k)}}{\chi(1)}.$$

Formula (6) follows immediately from this result and orthogonality relations (4).

Lemma 11 *For any irreducible character χ_ρ on G,*

$$\sum_{g \in G} \chi_\rho\left(h_1 g h_2 g^{-1}\right) = \frac{|G|\chi_\rho(h_1)\chi_\rho(h_2)}{\chi_\rho(1)}. \tag{7}$$

Proof :

$$\sum_{g \in G} \chi_\rho\left(h_1 g h_2 g^{-1}\right) = \frac{|G|}{|\mathrm{Cl}_G(h_2)|}\chi_\rho(h_1\widehat{\mathrm{Cl}_G(h_2)}) \tag{8}$$

but also

$$\sum_{g \in G} \chi_\rho\left(g^{-1} h_1 g h_2\right) = \frac{|G|}{|\mathrm{Cl}_G(h_1)|}\chi_\rho(\widehat{\mathrm{Cl}_G(h_1)}h_2) \tag{9}$$

From (8) and (9) one can see that $\sum_{g \in G} \chi_\rho\left(h_1 g h_2 g^{-1}\right)$ does not depend of $h_1 \in \mathrm{Cl}_G(h_1)$ or $h_2 \in \mathrm{Cl}_G(h_2)$, so

$$\sum_{g \in G} \chi_\rho\left(h_1 g h_2 g^{-1}\right) = \frac{|G|}{|\mathrm{Cl}_G(h_1)||\mathrm{Cl}_G(h_2)|}\chi_\rho(\widehat{\mathrm{Cl}_G(h_1)}\widehat{\mathrm{Cl}_G(h_2)}). \tag{10}$$

Equation (10) together with (6) gives the result.

We are now ready to prove Theorem 7.

Proof : [Thm 7] Without loss of generality we assume that W is the isotypic subspace associated to $\chi_1, ..., \chi_s$.

Let

$$B := \sum_{g \in G} \sum_{h_1, h_2 \in H} E(h_1) E(h_2) \chi_\rho \left(h_1 g h_2 g^{-1} \right)$$

Then,

$$B = \sum_{h_1, h_2 \in H} E(h_1) E(h_2) \sum_{g \in G} \chi_\rho \left(h_1 g h_2 g^{-1} \right)$$

Using formula (7) we have,

$$B = \sum_{h_1, h_2 \in H} E(h_1) E(h_2) \frac{|G| \chi_\rho(h_1) \chi_\rho(h_2)}{\chi_\rho(1)}.$$

Since $E(h) = \sum_{i=1}^{s} \chi_i(1) \overline{\chi_i(h)}$ and λ_i is the multiplicity of χ_i in the restriction of χ_ρ to H we can apply the orthogonality relation (4) to get

$$B = \frac{|G||H|^2 \left(\sum_{i=1}^{s} \lambda_i \chi_i(1) \right)^2}{\chi_\rho(1)}.$$

Let $m := \sum_{i=1}^{s} \lambda_i \chi_i(1)$ and $n := \chi_\rho(1)$ summing each side of (5) for all $g \in G$ gives :

$$(|G| - |H|) d_c^2 = |G| m - \frac{1}{|H|^2} B$$

hence

$$(|G| - |H|) d_c^2 = |G| m - \frac{|G| m^2}{n}$$

and

$$d_c^2 = \frac{|G|}{|G| - |H|} \left(m - \frac{m^2}{n} \right).$$

If $N := |G|/|H|$,

$$d_c^2 = \frac{N}{N-1} \frac{m(n-m)}{n}$$

Looking at the definitions of m, n and N we have exactly a packing of N elements in $\mathcal{G}_{m,n}$ which reaches the simplex bound.

4 Examples

This section is devoted to various examples coming from the classification of 2-transitive groups. We recall this classification to show that the previous construction gives infinite families of codes meeting the simplex bound, and to allow computation of examples not handled in this paper.

4.1 The classification of 2-transitive groups

Faithful 2-transitive groups have been classified [2, 3] (this result rely on the classification of simple group). This classification (achieved in the early 80's) has required work of various people (Huppert, Hering, Curtis, Kantor...). To summarize, there are eight types of infinite families of 2-transitive groups and some 'sporadic' groups. Among the eight families four are quite easy to describe:

1. The alternating groups : $\mathfrak{A}_n$ acting on the set $\{1, \ldots, n\}$ ($n-2$ transitive).

2. The symmetric groups : $\mathfrak{S}_n$ acting on the set $\{1, \ldots, n\}$ (n transitive).

3. Affine groups : Let V be the vector space $(\mathbb{F}_q)^d$. Affine groups are the groups $G := V \rtimes G_0$ where G_0 is a subgroup[9] of $\Gamma L_d(\mathbb{F}_q)$ which verifies one of the following conditions :

   - $\mathrm{SL}_d(\mathbb{F}_q) \leq G_0 \leq \Gamma L_d(\mathbb{F}_q)$,
   - $\mathrm{Sp}_d(\mathbb{F}_q) \leq G_0 \leq \Gamma L_d(\mathbb{F}_q)$ where $d = 2m$,
   - $G_0 = G_2(2^m)$.

   There is also a finite number of special cases with dimensions ≤ 6.

4. Projective groups : These are the groups G with $\mathrm{PSL}_d(\mathbb{F}_q) \leq G \leq \mathrm{P\Gamma L}_d(\mathbb{F}_q)$ acting on lines of $(\mathbb{F}_q)^d$.

There are four other families of 2-transitive groups coming from groups of Lie type. Describing these groups in details is not the aim of this article, one can use the ATLAS [23] to have a permutation representation or see the references for more details.

5. Symplectic groups : $G = Sp_d(\mathbb{F}_2)$ ($d = 2m$) acting on subsets of transvections. The degree is $2^{d-1}(2^d - 1)$ or $2^{d-1}(2^d + 1)$.

6. The unitary projective groups : $\mathrm{PSU}_3(q) \leq G \leq \mathrm{P\Gamma U}_3(q)$ acting on isotypic lines of a quadratic form or on points of a $S(2, q+1, q^3+1)$ Steiner system.

7. Suzuki groups : $Sz(q)$, ($q = 2^{2m+1}$), acting on points of a $S(3, q+1, q^2+1)$ Steiner system.

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[9] $\Gamma L_d(\mathbb{F}_q)$ is the group acting on V generated by $\mathrm{GL}_d(\mathbb{F}_q)$ and all field automorphisms, $\sigma : \mathbb{F}_q \to \mathbb{F}_q$ acting component-wise on elements of V.

8. <u>Ree groups</u> : $R(q)$, $(q = 3^{2m+1})$, acting on the points of a $S(2, q+1, q^3+1)$ Steiner system.

There are also some 'sporadic' 2-transitive groups with peculiar actions. They are (cited with their respective degree) : $(M_{11}, 11), (M_{11}, 12), (M_{22}, 22), (M_{23}, 23), (M_{24}, 24), (A_7, 15), (PSL(2, 11), 11), (PSL(2, 8), 28), (HS, 176), (Co_3, 276)$.

Alas the use of this classification is partially theoretical. Indeed explicit representations and character tables are not known for all these 2-transitive groups. But they are known for some infinite families (as for example $PGL_2(\mathbb{F}_q)$ or $PSL_2(\mathbb{F}_q)$ and their upper triangular subgroups or simply $\mathfrak{S}_{N-1} \subset \mathfrak{S}_N$) and then they give birth to explicit infinite families of optimal simplicial configurations. Next subsections are devoted to present the parameters obtained with these groups.

4.2 The symmetric group

Taking $G = \mathfrak{S}_N$ and $H = \mathrm{Stab}(1) \cong \mathfrak{S}_{N-1}$ (so $G/H \approx \{1, \ldots, N\}$), the action of G is 2-transitive on G/H. The irreducible representations of $\mathfrak{S}_N$ are well known (see [9]).

Table 1 gives the parameters of codes constructed from definition 4 and theorem 7 with $G = \mathfrak{S}_N$ and $\mathfrak{A}_N$, $4 \leq N \leq 8$. The chordal distance is derived from the simplex bound (2). In fact representations of the symmetric group are known well enough to give an explicit and general method to obtain these parameters. With this method one can easily compute the parameters for any N. We have described this method in details in Table 2 and have developed an example in parallel.

4.3 The groups $\mathbf{PGL_2(\mathbb{F}_q)}$ and $\mathbf{PSL_2(\mathbb{F}_q)}$

We have fully studied all the constructions obtained with $PGL_2(\mathbb{F}_q)$ and $PSL_2(\mathbb{F}_q)$ (q odd). In this case the subgroup H can be chosen as the subgroup of upper triangular matrices. Explicit descriptions of irreducible representations and characters of these groups can be found in [4, 6–8]. So we can give all the simplicial configurations coming from representations and characters of these groups. For all q such that $\mathbb{F}_q$ exists (*i.e.* q is a power of a prime) we have Grassmannian configurations with parameters as given in Table 3.

We give the value of $\tilde{d}$ when it is known. This value is difficult to evaluate in the cases where a "?" appear. Indeed numerical experiments show that depending on q this value can be 0. We were unable to make any conjecture about its behavior. For example the sets of principal angles corresponding to the last column configuration are given in Table 4 (we give the values of $\sin^2(\theta_i)$)

Table 1: Parameters for codes coming from $\mathfrak{S}_N$ and $\mathfrak{A}_N$, $4 \le N \le 8$

$N=4$	
n	3
m	1
d_c^2	$\frac{8}{9}$

$N=5$			
n	4	5	6
m	1	$1,2$	3
d_c^2	$\frac{15}{16}$	$1, \frac{3}{2}$	$\frac{15}{8}$

$N=6$					
n	5	8	9	10	16
m	1	3	4	4	$5,6$
d_c^2	$\frac{24}{25}$	$\frac{9}{4}$	$\frac{8}{3}$	$\frac{72}{25}$	$\frac{33}{8}, \frac{27}{4}$

$N=7$					
n	6	14	15	20	21	35
m	1	5	5	10	$5,8$	$8,9,10,16,17$
d_c^2	$\frac{35}{36}$	$\frac{15}{4}$	$\frac{35}{9}$	$\frac{35}{6}$	$\frac{40}{9}, \frac{52}{9}$	$\frac{35}{6}, \frac{39}{5}, \frac{25}{3}, \frac{152}{15}, \frac{51}{5}$

$N=8$										
n	7	20	21	28	35	42	45	56	64	70	90
m	1	6	6	14	$10,15$	21	10	21	$14,15,35$	$14,21,35$	$20,35$
d_c^2	$\frac{45}{49}$	$\frac{24}{5}$	$\frac{240}{49}$	8	$\frac{400}{49}, \frac{480}{49}$	12	$\frac{80}{9}$	15	$\frac{25}{2}, \frac{105}{8}, \frac{145}{8}$	$\frac{64}{5}, \frac{84}{5}, 20$	$\frac{160}{9}, \frac{220}{9}$

4.4   Some Symplectic and Sporadic groups

In the examples of previous sections, representations and characters are well known. So an explicit description of the codes (by projection matrices for instance) is possible and the values of principal angles can be given. This may not be the case for codes coming from high degree irreducible representations of big symplectic groups or simple groups (such as $Sp_{10}(2)$ or Co_3). Nevertheless one may be interested in small parameters coming from such groups given in Tables 5 and 6. The chordal distance is recorded when its rational form involves coefficients with few digits. In any case it can be derived from the simplex bound (2).

To compute these tables we used permutations to describe these groups. The degree of the permutation representations[10] was chosen according to the classification of 2-transitive groups (see section 4.1). Then the subgroup H can be chosen as $\mathrm{Stab}(1, G)$.

5   Generalizations

In this section we discuss various generalizations of the previous result. First we give a method to expand the set of codes reaching the simplex bound. Then we have a look at unions of configurations coming from group orbits. Finally

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[10]These descriptions of the Symplectic and Sporadic groups can be found in the ATLAS [23].
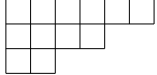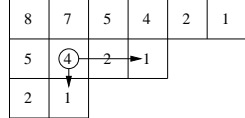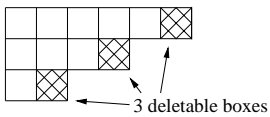
Table 2: The example of $\mathfrak{S}_N$

Facts	Example
• There is an irreducible representation of $\mathfrak{S}_N$ for each partition $\lambda = [\lambda_1, \ldots, \lambda_k]$ of n (*i.e.* a decreasing sequence of integers $[\lambda_1, \ldots, \lambda_k]$ whose sum is N).	N:=12 $\lambda := [6, 4, 2]$
• One can associate a diagram to a partition in the following way. Draw λ_1 box on the first line, λ_2 on the second... As in the example.	
• The hook length of a box is the sum of the number of boxes under it (in the same column) and at its right (in the same line) plus one. In the example, we have filled each box with the length of the associated hook.	
• The dimension of the representation associated to λ is given by $\frac{N!}{z}$ where z is the product of the hook length of every box.	$\dim(\chi_\lambda) = 2673$
• The branching rule (see [9]) states that when restricted to $\mathfrak{S}_{N-1}$ the irreducible character associated to lambda decomposes itself as $\chi_\lambda = \chi_{\mu_{(1)}} + \cdots + \chi_{\mu_{(\ell)}}$ where each $\mu_{(i)}$ is obtained from λ by deleting a 'corner' box.	3 deletable boxes $\chi_{[6,4,2]} \downarrow_{\mathfrak{S}_{N-1}}^{\mathfrak{S}_N} =$ $\chi_{[5,4,2]} + \chi_{[6,3,2]} + \chi_{[6,4,1]}$
• Now we have everything needed to compute the parameters obtained. Let us choose the isotypic space associated to $\chi_{[5,4,2]}$. Its dimension can be computed with the hook formula and have a configuration with the following parameters.	$N = 12,\ n = 2673,\ m = 990$ so $d_c^2 = 680$ (reaching the Simplex bound)

13

Table 3: Explicit parameters coming from $\mathrm{PGL}_2(\mathbb{F}_q)$ and $\mathrm{PSL}_2(\mathbb{F}_q)$ (q odd)

$$N := q+1$$

n	$q+1$	q	$q+1$	q
m	1	1	2	$\frac{q-1}{2}$
d_c^2	1	$1 - \frac{1}{q^2}$	$2\frac{q-1}{q}$	$\frac{(q+1)^2(q-1)}{4q^2}$
$\tilde{d}$	1	$1 - \frac{1}{q^2}$	$\left(\frac{q-1}{q}\right)^2$	$?$

n	$q+1$	$q+1$	$\frac{q+1}{2}$	$q-1$
m	$\frac{q-1}{2}$	$\frac{q+1}{2}$	1	$\frac{q-1}{2}$
d_c^2	$\frac{(q-1).(q+3)}{4.q}$	$\frac{(q+1)^2}{4.q}$	$\frac{q-1}{q}$	$\frac{q^2-1}{4.q}$
$\tilde{d}$	$?$	$?$	$\frac{q-1}{q}$	$?$

Table 4: Explicit values of principal angles

$n=q-1,\ m=\frac{q-1}{2}$ $N=q+1,\ d_c^2=\frac{q^2-1}{4q}$	
$q=5$	$\left(1, \frac{1}{5}\right)$
$q=7$	$\left(0, \frac{6}{7}, \frac{6}{7}\right)$
$q=9$	$\left(1, \frac{5}{9}, \frac{1}{3}, \frac{1}{3}\right)$
$q=11$	$\left(0, \underbrace{\frac{7+3\sqrt{5}}{22}, \frac{7-3\sqrt{5}}{22}}_{\text{twice}}\right)$

Table 5: Small parameters coming from small symplectic groups

$Sp_4(2)$ $\quad N = 10$			
n	5	8	9	10
m	1	4	$1,4$	$1,2,4,5$
d_c^2	$\frac{8}{9}$	$\frac{20}{9}$	$\frac{80}{81}, \frac{320}{81}$	$1, \frac{16}{9}, \frac{8}{3}, \frac{25}{9}$

$Sp_4(2)$ $\quad N = 6$			
n	5	8	9	10
m	1	3	4	$3,4$
d_c^2	$\frac{24}{25}$	$\frac{9}{4}$	$\frac{8}{3}$	$\frac{63}{25}, \frac{72}{25}$

$Sp_6(2)$ $\quad N = 36$ $\quad n \le 100$						
n	15	21	27	35	56	70	84
m	1	1	7	$1,14,15$	$7,21,28$	28	$14,28$
d_c^2	$\frac{24}{25}$	$\frac{48}{49}$	$\frac{16}{3}$	$\frac{1224}{1225}, \frac{216}{25}, \frac{432}{49}$	$\frac{63}{10}, \frac{27}{2}, \frac{72}{5}$	$\frac{432}{25}$	$12, \frac{96}{5}$

$Sp_6(2)$ $\quad N = 28$ $\quad n \le 100$						
n	7	21	27	35	56	70	84
m	1	$1,6$	$1,6,7$	15	$6,20,26$	10	24
d_c^2	$\frac{8}{9}$	$\frac{80}{81}, \frac{40}{9}$	$\frac{728}{729}, \frac{392}{81}, \frac{3920}{729}$	$\frac{80}{9}$	$\frac{50}{9}, \frac{40}{9}, \frac{130}{9}$	$\frac{8}{9}$	$\frac{160}{9}$

$Sp_8(2)$ $\quad N = 136$ $\quad n \le 1000$							
n	51	85	119	135	238	510	595	918
m	1	1	35	$1,50,51$	28	210	$28,175$	$50,168,218$
d_c^2	$\frac{80}{81}$	$\frac{224}{225}$	$\frac{224}{9}$	...	$\frac{224}{9}$	$\frac{1120}{9}$	$\frac{627}{25}, \frac{1120}{9}$	...

$Sp_8(2)$ $\quad N = 120$ $\quad n \le 1000$							
n	35	85	119	135	238	510	595	918
m	1	1	$1,34,35$	51	34	34	$34,204,238$	204
d_c^2	$\frac{48}{49}$	$\frac{288}{289}$	$\frac{944}{945}, \frac{4624}{189}, \frac{224}{9}$	...	$\frac{9248}{315}$	...	...	...

$Sp_{10}(2)$ $\quad N = 528$ $\quad n \le 10000$				
n	187	341	495	527	6138
m	1	1	155	$1,186,187$	868
d_c^2	$\frac{288}{289}$	$\frac{960}{961}$	$\frac{320}{3}$	...	$\frac{2240}{3}$

$Sp_{10}(2)$ $\quad N = 496$ $\quad n \le 10000$				
n	155	341	495	527	6138
m	1	1	$1,154,155$	187	$154,748,902$
d_c^2	$\frac{224}{225}$	$\frac{1088}{1089}$	...	$\frac{1088}{9}$	...

Table 6: Small parameters coming from some sporadic groups

Co_3 $N = 276$								
n	23	253	275	1771	2024	4025	5544	7084	...
m	1	$1, 22$	$1, 22, 23$	231	$22, 252, 274$	22,231,252,253, 274,483,505	22,252,274,1750, 1772,2002,2024	1540	...
d_c^2	$\frac{24}{25}$	$\frac{3024}{3025}, \frac{504}{25}$	...	$\frac{1008}{5}$	...	...	...	$\frac{6048}{5}$	...

HS $N = 176$						
n	22	77	154	175	231	...	3200
m	1	21	$1, 21, 28, 29, 49$	$1, 21, 22, 28, 29, 49, 50$	$21, 84, 105$	...	more than 200 possibilities
d_c^2	$\frac{24}{25}$	$\frac{384}{25}$	$\frac{1224}{1225}, \frac{456}{25}, \frac{576}{25}, \frac{1160}{49}, \frac{168}{5}$	...	$\frac{96}{5}, \frac{1344}{25}, \frac{288}{5}$	...	...

M_{24} $N = 24$							
n	23	252	483	1035	1265	1771	2277	...
m	1	22	230	45	230	$231, 770$	253	...
d_c^2	$\frac{528}{529}$	$\frac{440}{21}$	$\frac{880}{7}$	...	$\frac{2160}{11}$	...	$\frac{704}{3}$	...

we give a new insight on some optimal orthoplex configurations given in [15].

5.1   How to extend the set of parameters

We have already found a large set of parameters for codes reaching the simplex bound but it may be interesting to extend it further. Thinking in terms of projection matrices one can have the idea to use Kroenecker product ($\otimes$). As a first try we can make the product of all matrices of a configuration with the identity matrix of rank k : I_k.

$$\{I_k \otimes \Pi_{g.W} \; : \; g \in G/H\}$$

One can easily see that this multiplies m, n, d_c^2 by k and keeps N invariant. So we have the following :

Proposition 12 *If we have an explicit configuration with parameter N, m, n which reach the simplex bound, then for any positive integer k it is possible to build a new optimal configuration with parameters $N, k.m, k.n$.*

This idea can be extended to couple of configuration. Let $\{\Pi_{1,i} \; : \; i \in [\![1, \dots, N_1]\!]\}$ and $\{\Pi_{2,j} \; : \; j \in [\![1, \dots, N_2]\!]\}$ be two configurations in G_{m_1,n_1} and G_{m_2,n_2} with squared minimal distances equal to d_{c1}^2 and d_{c2}^2. The eigenvalues of $(\Pi_1 \otimes \Pi_2)(\Pi_1' \otimes \Pi_2')$ are the products $\lambda_i \mu_j$ where λ_i and μ_j are the eigenvalues of $\Pi_1 \Pi_1'$ and $\Pi_2 \Pi_2'$ respectively. One can then deduce that the new chordal distance is $\min(m_1 d_{c2}^2, m_2 d_{c1}^2)$.

5.2 Minimal distance in unions

In this section we study in unions of configurations obtained by orbits of isotypic subspaces. We first state a variation of proposition 5 (with a similar proof).

Proposition 13 *Let W_1 and W_2 be two subspaces of $\mathbb{C}^n$ stable under the action of H. The number of different sets of principal angles between $g_1 W_1$ and $g_2 W_2$ (g_1, $g_2 \in G$) is bounded above by the number of orbits for the action of G on ordered pairs of G/H. Moreover the set of principal angles for the pairs $(W_1, g_0 W_2)$ and $(W_1, g W_2)$ is the same for all $g \in H g_0 H$.*

This property can be used to find the minimal chordal distance in unions of orbits. We can for example use it with configurations coming from 2-transitive groups, this gives the following theorem.

Theorem 14 *Let G be a group and H a subgroup such that G act 2-transitively by left multiplication on G/H. Suppose furthermore that we dispose of an irreducible representation $\rho : G \to GL_n(\mathbb{C})$ which is reducible when restricted to H. If $W_1, ..., W_t$ are direct sums of H-isotypic subspaces associated to disjoints subsets of irreducible characters, all W_i's having the same dimension m, then the orbit of $W_1, ..., W_t$ under G gives a code of cardinal $N = t|G/H|$ having minimal distance $d_c^2 = \frac{N}{N-1} \frac{m(n-m-n/N)}{n}$.*

Proof : Suppose that $\chi_\rho \downarrow_H^G = \sum_{i=1}^s \lambda_i.\chi_i$, and that we have found disjoints subsets $S_1, \ldots, S_t$ of $[\![1, \ldots, s]\!]$ such that

$$\forall j \in [\![1, \ldots, t]\!], \ \sum_{i \in S_j} \lambda_i.\chi_i(1) = m$$

for a fixed m.

We need to compute the distance value for any pair of m dimensional planes. Let W_1 and W_2 be the direct sums of isotopic subspaces associated to S_1 and S_2 and focus on the distance between W_1 and $g.W_2$.

If $g \in H$ then $g.W_2 = W_2$ and the two spaces W_1 and W_2 are orthogonal so the minimal distance between them is m. If $g \notin H$ the chordal distance may take only one value by Proposition 13.

If we sum over $g \in G$ the analog of (5) with different characters we get the following equality :

$$|H|m + (|G| - |H|)d_c^2(W_1, gW_2) = |G|m - \frac{1}{|H|^2}B$$

where

$$B = \sum_{g \in G} \sum_{h_1, h_2 \in H} E_{S_1}(h_1) E_{S_2}(h_2) \chi_\rho(h_1 g h_2 g^{-1})$$

and

$$E_{S_j}(h) := \left(\sum_{i \in S_j} \chi_i(1)\overline{\chi_i(h)} \right)$$

Following the proof of Theorem 7, we found that

$$B = \frac{|G||H|^2 \left(\sum_{i \in S_1} \lambda_i \chi_i(1) \right) \left(\sum_{i \in S_2} \lambda_i \chi_i(1) \right)}{\chi_G(1)}$$

so

$$d_c^2(W_1, gW_2) = \frac{N}{N-1} \frac{m(n - m - \frac{n}{N})}{n}.$$

This value is smaller than $d_c^2(W_1, W_2) = m$ or $d_c^2(W_1, gW_1)$. It is therefore the minimal chordal distance.

We may carry on the example of Table 1. One can observe that the two components $\chi_{[5,4,2]}$ and $\chi_{[6,3,2]}$ in the decomposition of $\chi_{[6,4,2]} \downarrow_{\mathfrak{S}_{11}}^{\mathfrak{S}_{12}}$ have the same dimension (990). So if we take the union of the two configurations we get a configuration of $N = 24$ planes of dimension $m = 990$ in $\mathbb{C}^{2673}$ with minimal distance $d_c^2 = \frac{13970}{23} \approx 607.4$ (for a value of 650.4 of the simplex bound). This may lead to not so bad configurations especially if we have a lot of subspaces with the same dimension (for the new value of minimal distance does not depend on how many sets are joined).

5.3   An optimal orthoplex configuration

In this section we show how a small variation of the method described above gives back the optimal configurations presented in [15] (Thm 1). We first recall the construction of the keystone groups :

Let $U = \mathbb{F}_2^i$ and $V = \mathbb{R}^n$ where $n = 2^i$. Let $\{e_u : u \in U\}$ be a vector basis for V and let E be the (extraspecial) subgroup of the orthogonal group $\mathcal{O} = O(V)$ generated by

$$X(a) : e_u \mapsto e_{u+a}, \text{ and } Y(b) : e_u \mapsto (-1)^{b.u} e_u, \quad u \in U.$$

The normalizer L of E in $\mathcal{O}$ is the (Clifford type) subgroup of $\mathcal{O}$ generated by[11]: E, H, $\tilde{H}_2$, $\widetilde{GL(V)}$ and $\{d_M : M \text{ skew-symmetric}\}$, where

- $H = \frac{1}{\sqrt{N}}[(-1)^{u.v}]_{u,v \in V}$

- $\tilde{H}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes I_{2^{(i-1)}}$

---

[11]See [16].

- $\widetilde{GL(V)}$ is the group generated by the orthogonal transformations $\{G_A : A \in \mathrm{GL}(U)\}$ where $G_A : V \to V$ permute the coordinates by sending e_u on e_{Au}.

- d_M is the diagonal matrix $(-1)^{Q_M(v)}$ where Q_M is the quadratic form associated to the skew-symmetric matrix M i.e. $Q_M(u+v) = Q_M(u) + Q_M(v) + uMv^T$.

Note that this description of the group E is a unitary representation and its character χ_E as value 0 for any element except $\pm I$.

Let $\mathcal{S}_r$ be the set of abelian subgroups in E generated by $-I$ and r independent order 2 element $g_1, ..., g_r$ of E and $S \in \mathcal{S}_r$. Then the restriction of χ_E to S is equal to the sum of 2^r distinct linear characters with multiplicity 2^{i-r}.

The configuration described in [15] (Thm 1) is the set of all isotypic subspaces for all subgroups S in $\mathcal{S}_r$ and all their characters.

Fix $S \in \mathcal{S}_r$ and χ an irreducible character and let $\Pi_{S,\chi} = \frac{1}{|S|} \sum_{s \in S} \overline{\chi(s)} s$ be the projection on the associated isotypic subspace W. Considering the orbit of W under L (instead of E), one can check that

$$
\left\{ \rho(g)\Pi_{S,\chi}\rho(g^{-1}) = \frac{1}{|S|} \sum_{s \in gSg^{-1}} \overline{\chi(g^{-1}sg)} s \quad : \quad g \in L \right\}
$$

is exactly the set of projection matrices of the the Grassmannian code described in [15] (Thm 1). This is a consequence of the following facts :
- the action of L by conjugation on $\mathcal{S}_r$ is transitive ;
- the action of E on characters of S given by $g.\chi(x) = \chi(gxg^{-1})$ is transitive.

Let now focus on the optimal case where $r = 1$ (i.e. $|S| = 4$), the optimality of this configuration is a special case of the following proposition :

Proposition 15 *Let G be a subgroup of $U_n(\mathbb{C})$ such that the only elements having a non-zero trace are $\pm I$. Let $\mathcal{S}$ be the set of distinct abelian subgroups generated by $-I$ and another element g of order 2. Then the set of isotypic subspaces with projection matrices*

$$
\frac{1}{|S|} \sum_{s \in S} \chi_1(s)s \quad and \quad \frac{1}{|S|} \sum_{s \in S} \chi_2(s)s \quad for\ S \in \mathcal{S}
$$

where

$$
\chi_i(-1) = -1 \quad and \quad \chi_i(g) = (-1)^i
$$

form a Grassmannian code in $\mathcal{G}_{m,n}$ with $m = \frac{n}{2}$ and cardinality $2|\mathcal{S}|$ in which the only non-zero distances are m and $m/2$. This code reaches the orthoplex bound when $2|\mathcal{S}| > \frac{n(n+1)}{2}$.

19

Proof : For convenience we suppose that the above group is obtained by a representation $\rho : G \to U_n(\mathbb{C})$. We first remark that $W = \frac{1}{|S|} \sum_{s \in S} \chi_i(s)s$ is an isotypic subspace. The restriction of the matrix representation to S is reducible. It is clear that its character decomposition is

$$\chi_\rho \downarrow_S^G = \frac{n}{2}\chi_1 + \frac{n}{2}\chi_2.$$

So the isotypic subspaces have dimension $m = \frac{n}{2}$ (and n is divisible by 2). Let $W_1 := \frac{1}{|S_1|} \sum_{s \in S_1} \chi(s)s$ and $W_2 := \frac{1}{|S_2|} \sum_{s \in S_2} \chi'(s)s$ be two subspaces defined by the subgroups S_1 and S_2 in $\mathcal{S}$, and the irreducible characters χ and χ'. If the subgroups are equal and the characters distinct then the two isotypic subspaces are orthogonal and the chordal distance between W_1 and W_2 is equal to m. Otherwise

$$d_c^2(W_1, W_2) = m - \frac{1}{|S_1||S_2|} \sum_{\substack{s_1 \in S_1 \\ s_2 \in S_2}} \chi(s_1)\chi'(s_2)\chi_\rho(s_1 s_2).$$

Since $\chi_\rho(s_1 s_2)$ is not zero if and only if s_1 and s_2 are in $\{\pm I\}$ we have.

$$d_c^2(W_1, W_2) = m - \frac{1}{16} \sum_{s_1, s_2 \in \{\pm I\}} \chi(s_1)\chi'(s_2)\chi_\rho(s_1 s_2).$$

Hence

$$d_c^2(W_1, W_2) = m - \frac{4}{16}n = \frac{m}{2}.$$

Since the expression of the orthoplex bound (valid only if $N > \frac{n(n+1)}{2}$) is

$$d_c^2 \leq \frac{m(n-m)}{n} \tag{11}$$

it is clear that the above construction gives an optimal Grassmannian configuration as soon as $2|\mathcal{S}| > \frac{n(n+1)}{2}$.

6 Conclusion

In this article we have studied some interesting Grassmannian packings obtained as group orbits. We proved that orbits of isotypic subspaces associated to maximal subgroups of 2-transitive groups are optimal Grassmannian configurations w.r.t. the chordal distance. Indeed these configurations reach the simplex bound. Based on the classification of 2-transitive groups we have illustrated this result with many examples - among which some infinite families - for which we have computed the parameters. We have also recovered optimal configurations of [15] in a more general context. If our configurations perform well regarding the chordal distance they have a less obvious behavior regarding the pseudo-distance $\tilde{d}$. According to [18] theses codes are accurate for low rate communications over extremely noisy channels.

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References

[1] J. -C. Belfiore and A. M Cipriano, *Space-Time Wireless Systems: From Array Processing to MIMO Communications*, Cambridge University Press, 2006

[2] P. J. Cameron, *Permutation groups*, London Mathematical Society Student Texts, vol. 45, Cambridge University Press, 1999.

[3] J. D. Dixon and B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics, vol. 163, Springer-Verlag, 1996.

[4] W. Fulton, and J. Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, 1991.

[5] I. M. Isaacs, *Character theory of finite groups*, Corrected reprint of the 1976 original [Academic Press, New York], Dover Publications Inc., 1994.

[6] G. James, Gordon and M. Liebeck, *Representations and characters of groups*, Second Ed. Cambridge University Press, 2001.

[7] M. A. Naĭmark, A. I. Štern, *Theory of group representations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 246, Springer-Verlag, 1982.

[8] I. Piatetski-Shapiro, *Complex representations of* $GL(2, K)$ *for finite fields* K, Contemporary Mathematics, vol. 16, American Mathematical Society, 1983.

[9] B. E. Sagan, *The symmetric group*, Graduate Texts in Mathematics, vol. 203, Second Ed. Springer-Verlag, 2001.

[10] J.-P. Serre, *Représentations linéaires des groupes finis*, revised ED. , Hermann Paris, 1978.

[11] A. Terras, *Fourier analysis on finite groups and applications*, London Mathematical Society Student Texts, vol. 43, Cambridge University Press, 1999.

[12] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge University Press, 2005.

[13] I. Telatar, "Capacity of multi-antenna Gaussian channels", *European Trans. Telecommun.*, vol. 10, pp. 585–595, Nov./Dec. 1999.

[14] T. Marzetta and B. Hochwald, "Capacity of mobile multiple-antenna communication link in a Rayleigh flat-fading environment,", *IEEE Trans. Inform. Theory*, vol. 45, pp. 139–157, Jan. 1999.

[15] A. R. Calderbank, R. H. Hardin, E. M. Rains, P. W. Shor, N. J. A. Sloane "A group-theoretic framework for the construction of packings in Grassmannian spaces", *J. Algebraic Combin.*, vol. 9, no2, pp. 129–140, 1999.

[16] A. R. Calderbank, P. J. Cameron, W. M. Kantor and J. J. Seidel "Z_4-Kerdock codes, orthogonal spreads, and extremal Euclidean line-sets" *Proc. London Math. Soc. (3)*, vol. 75, no2, pp. 436–480, 1997.

[17] J. H. Conway, R. H. Hardin, and N. J. A. Sloane, "Packing lines, planes, etc.: packings in Grassmannian spaces", *Experiment. Math.*, vol. 5, no. 2, pp. 139–159, 1996.

[18] G. Han and J. Rosenthal, "Geometrical and numerical design of structured unitary space-time constellations" *IEEE Trans. Inform. Theory*, vol. 52, no. 8, pp. 3722–3735, 2006.

[19] JR. Hammons, A. Roger and H. El Gamal, "On the theory of space-time codes for PSK modulation", *IEEE Trans. Inform. Theory*, vol. 46, no. 2, pp. 524–542, 2000.

[20] A. Shokrollahi, B. Hassibi, B. M. Hochwald and W. Sweldens, "Representation theory for high-rate multiple-antenna code design", *IEEE Trans. Inform. Theory*, vol. 47, no. 6, pp. 2335–2367, Sep. 2001.

[21] V. Tarokh, N. Seshadri, and A. R. Calderbank "Space-time codes for high data rate wireless communication: performance criterion and code construction", *IEEE Trans. Inform. Theory*, vol. 44, no. 2, pp. 744–765, Mar. 1998.

[22] L. Zheng and D. N. C. Tse, "Communication on the Grassmann manifold: a geometric approach to the noncoherent multiple-antenna channel", *IEEE Trans. Inform. Theory*, vol. 48, no. 2, pp. 359–383, 2002.

[23] "The ATLAS of Finite Group Representations", available at : *http://brauer.maths.qmul.ac.uk/Atlas/*.